Topology: Basic ideas

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Historical background

- Topology appeared in late XIX – early XX century

  H. Poincare (1895)

  F. Hausdorff (1914)

- XIX century – geometry century

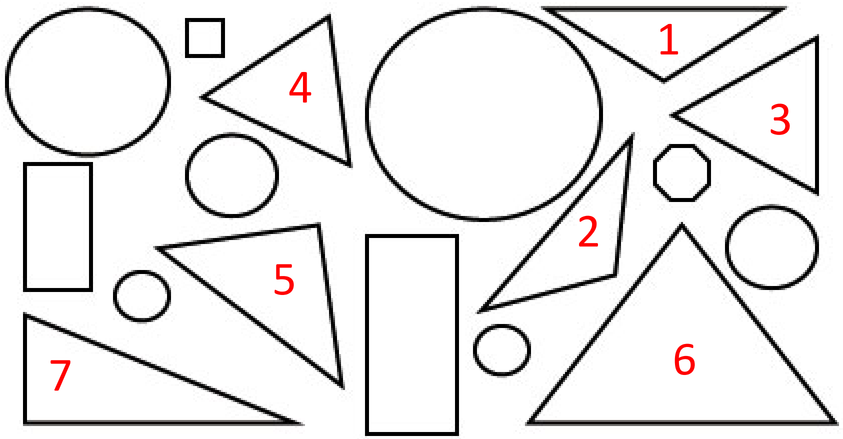
  N. Lobachevsky (1826)

  J. Boliai (1829)

  K. F. Gauss (1830s)

- Erlangen Program, F. Klein (1872)

What defines geometric properties?



Geometric characteristics: angles, lengths, etc.

$1 = 2, 3 = 4$

$1 \neq 4, 5 \neq 6, 5 \neq 7$, etc.

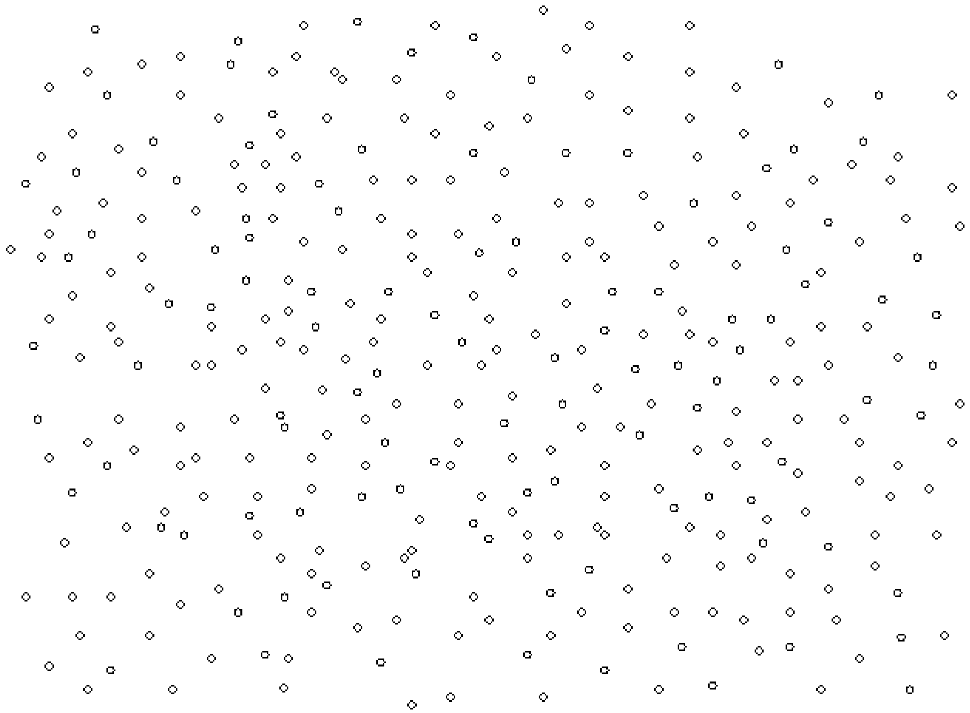
- How to match figures in Euclidean geometry: use translations + rotations
- Other geometries – other transformations, other geometric characteristics
- **The larger the group of transformations, the fewer geometric properties**
- **What is the “largest” group of transformations?**

Continuous transformations

What is the "largest" group of transformations that does not violate the structure of the space itself?

What is space? What is spatial continuity?

The fabric of space

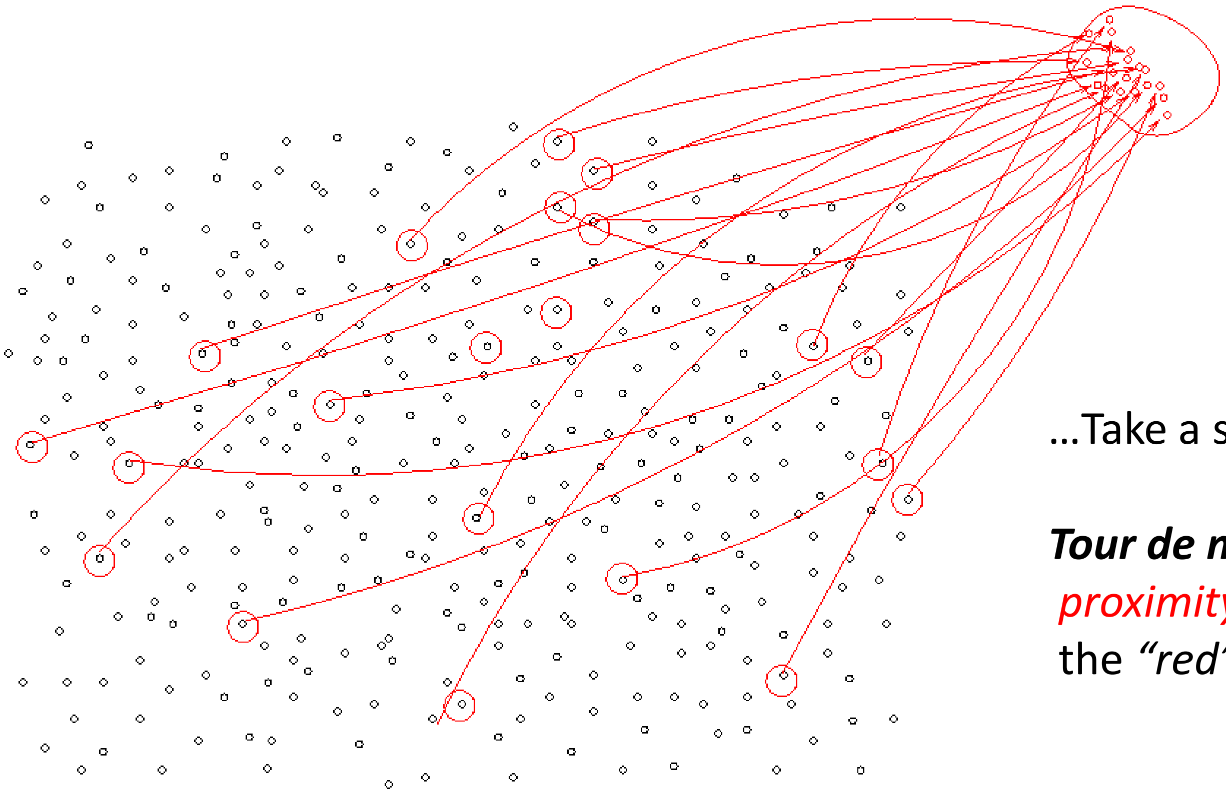


1. Space consists of points

2. What turns a set of points into a space?

3. ***<u>The notion of spatial proximity</u>***

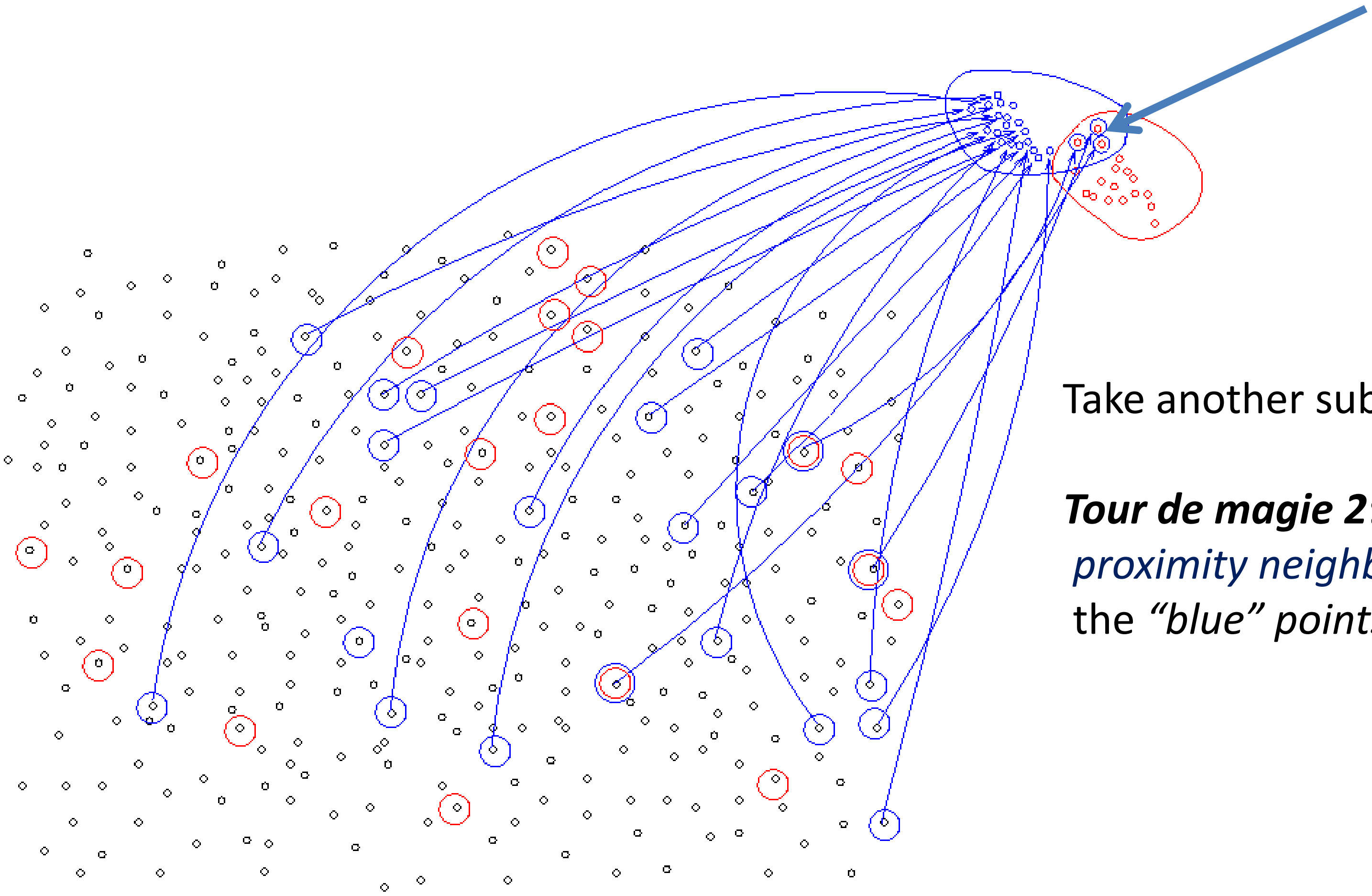
The fabric of space, constructively



…Take a subset of points

Tour de magie: define this set as a *proximity neighborhood*. From now on the *"red" points are neighbors*

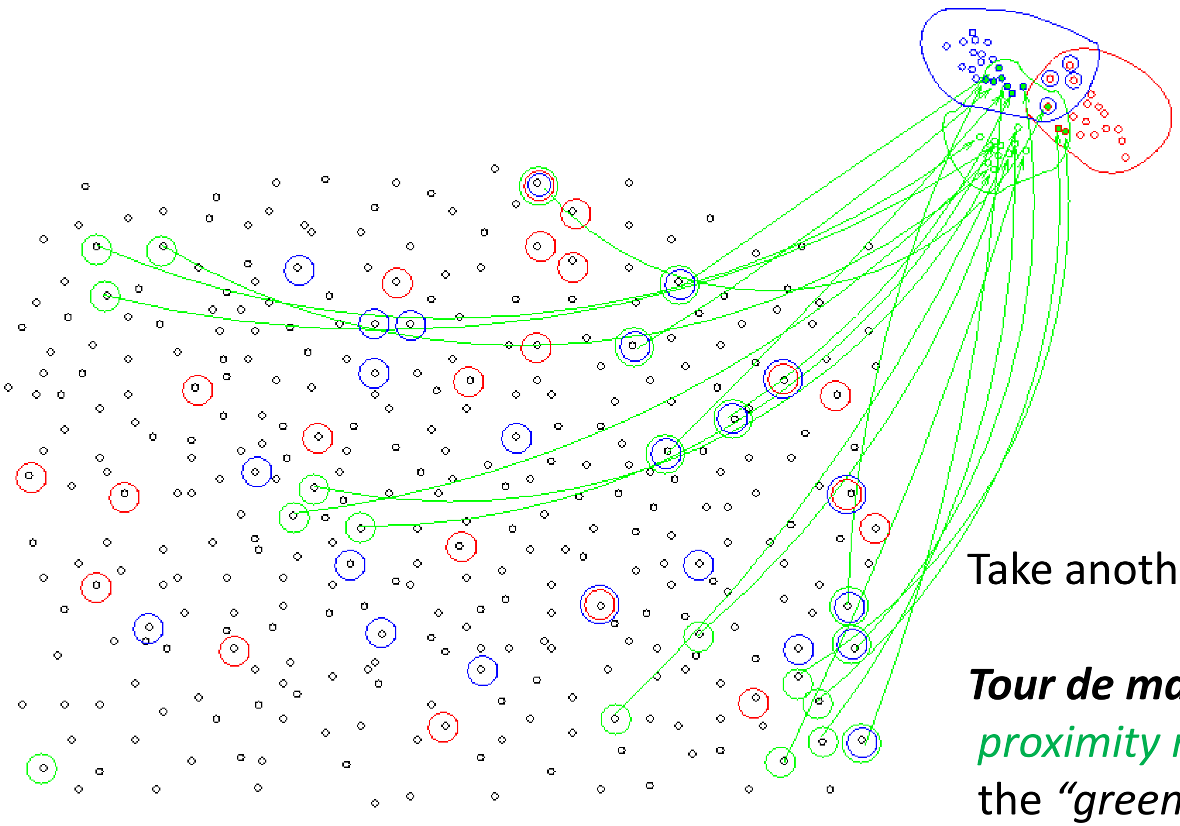
The fabric of space, constructively



Take another subset of points

Tour de magie 2: define this set as a *proximity neighborhood*. From now on the *"blue" points are also neighbors*

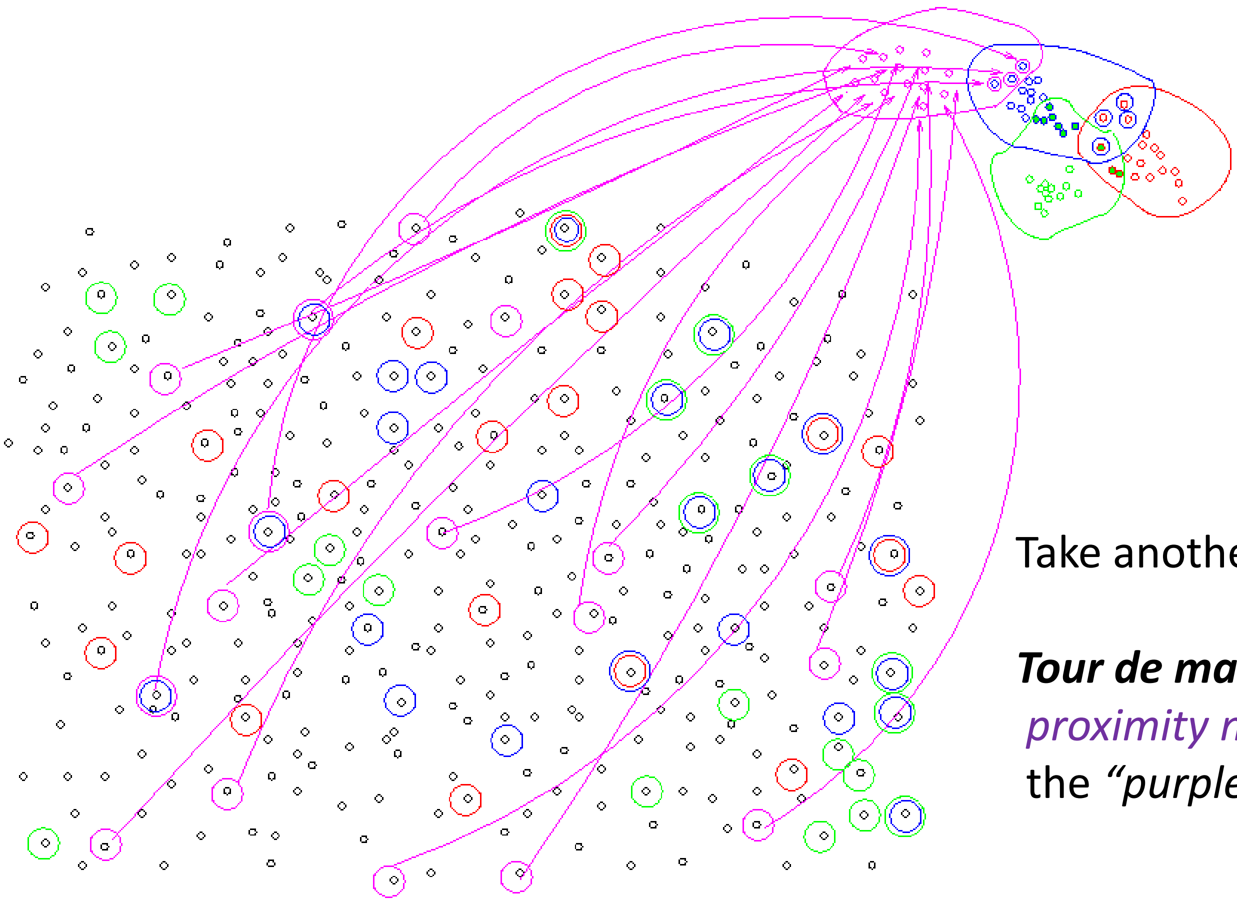
The fabric of space, constructively



Take another subset of points

Tour de magie 3: define this set as another *proximity neighborhood*. From now on the *“green” points are also neighbors*

The fabric of space, constructively



Take another subset of points

Tour de magie 4: define this set as another *proximity neighborhood*. From now on the *"purple" points are also neighbors*

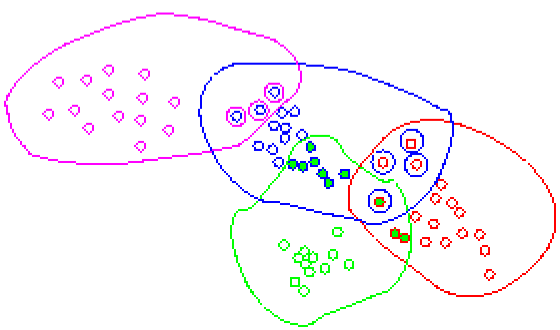
Topology, definition

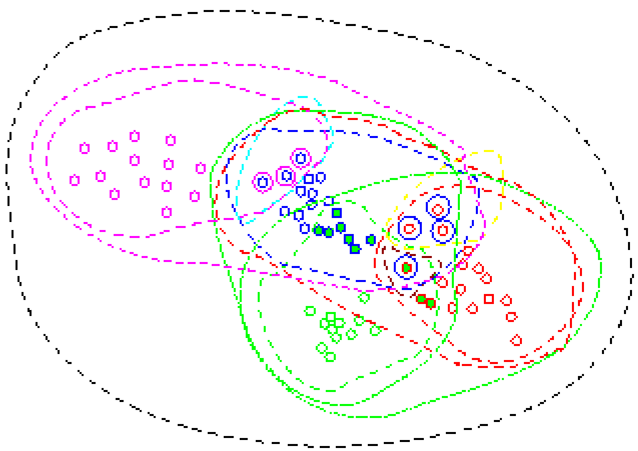
A set ***X*** with a system of neighborhoods, $T = \bigcup_i N_i$, such that:

1. The whole set ***X*** and the empty set are neighborhoods, $X \in T, \varnothing \in T$

2. A finite intersection of neighborhoods is a neighborhood, $\bigcap_{i=1}^{n} N_i \in T$

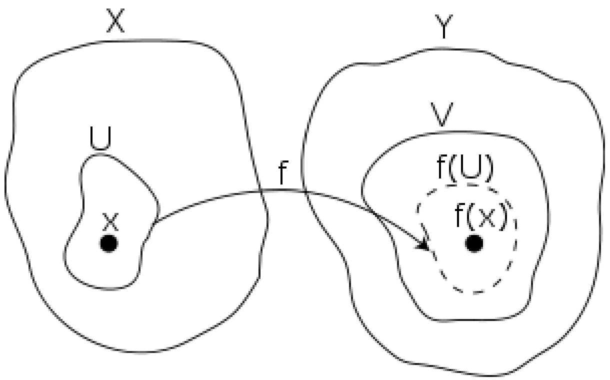
3. Any union of neighborhoods is a neighborhood $\bigcup_{i \in I} N_i \in T$





Continuous maps

Continuous maps: maps that preserve topological structure



Topology: area of mathematics concerned with spatial properties that are preserved under ***continuous*** deformations of spaces and geometric objects

Intuitively: elastic stretches and deformations are allowed, cutting and gluing is not

Topological properties, examples

What are the topological properties ?

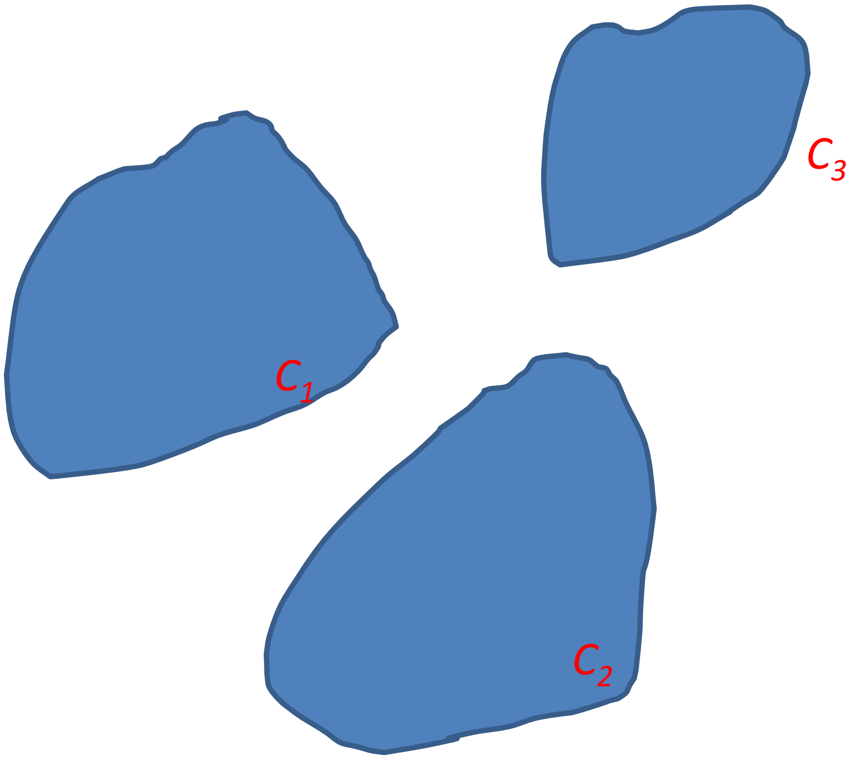
How to define them?

How to describe them?

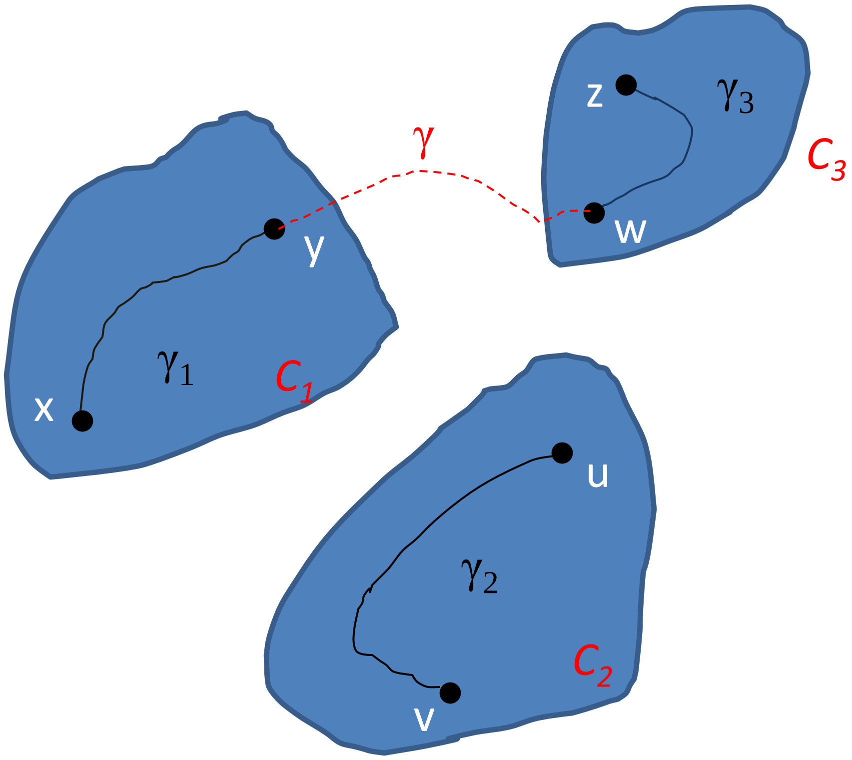
How to compute them?

Topological properties, examples

A space can have many pieces

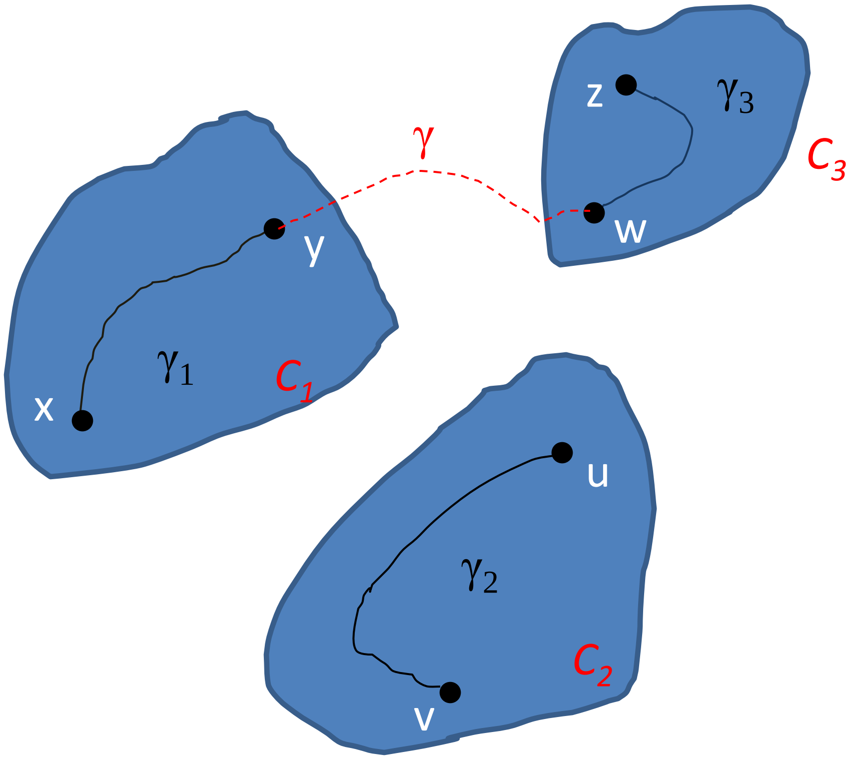


Topological properties, examples



Path connectedness

Topological properties, examples

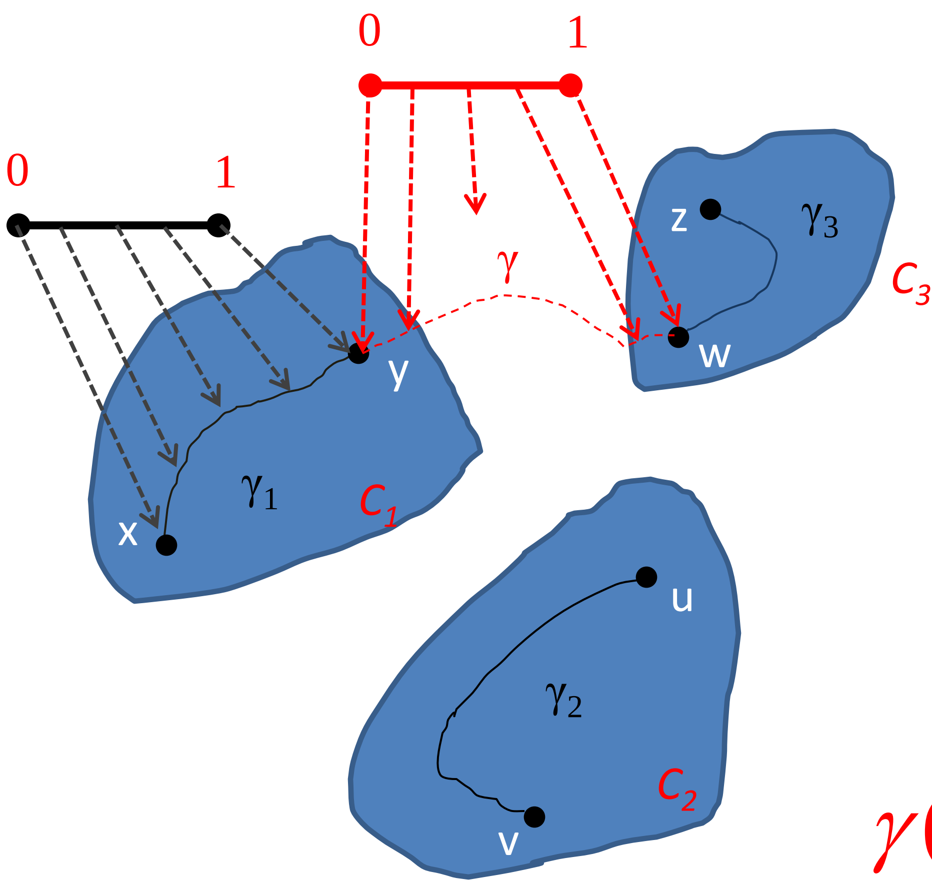


Path:

$$I = [0,1] \rightarrow \gamma(t)$$

Path connectedness

Topological properties, examples



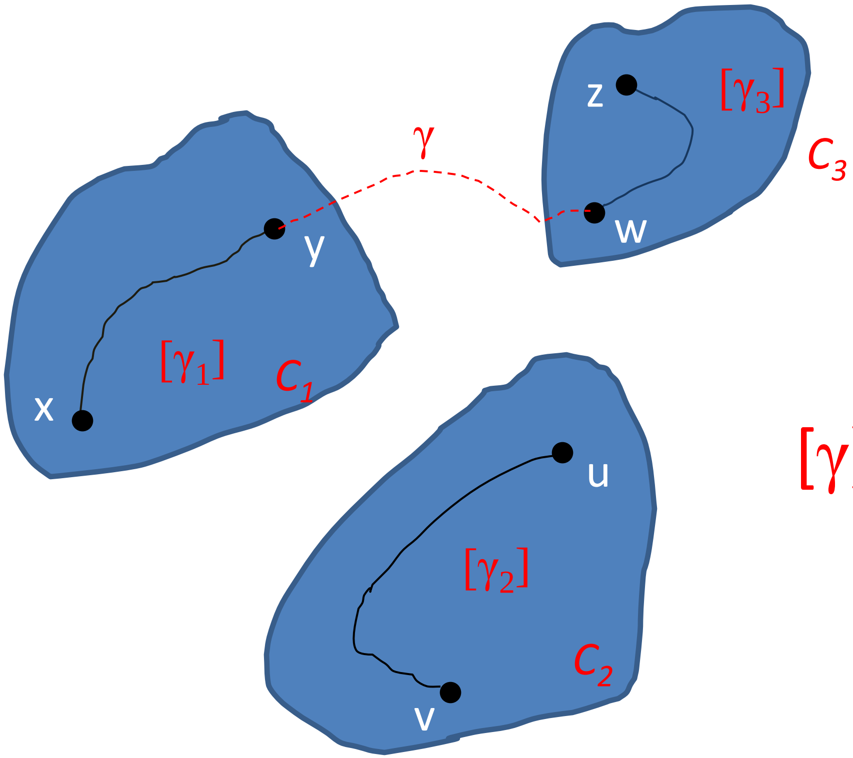
Path:

$$I = [0,1] \to \gamma(t)$$

$$\gamma(0) = y,\ \gamma(1) = w\ \ldots?$$

Path connectedness

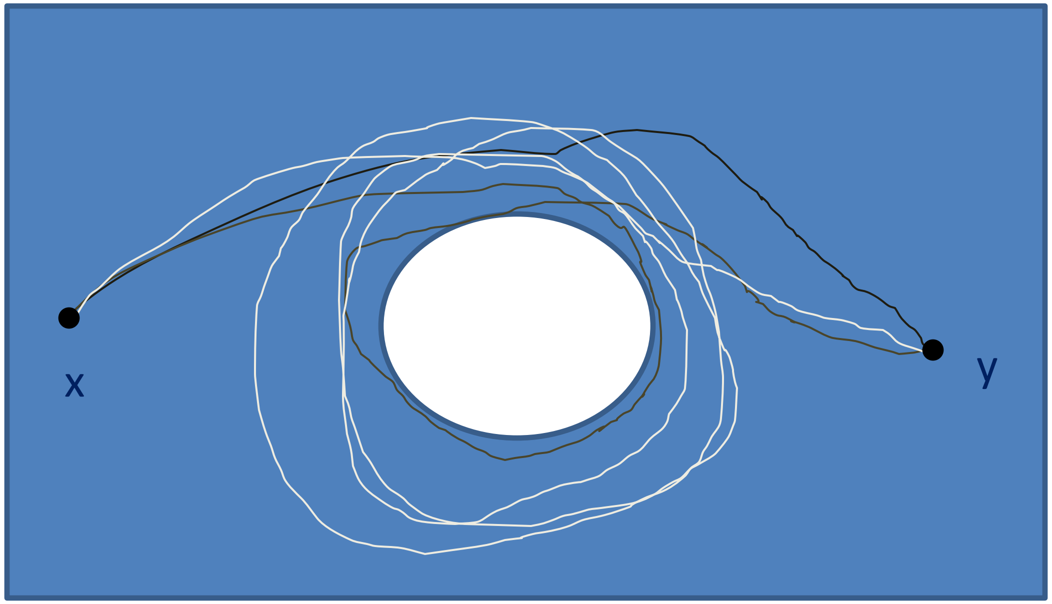
Topological properties, examples



$[\gamma]_1 \in C_1$, $[\gamma]_2 \in C_2$, $[\gamma]_3 \in C_3$.

Path connectedness: there are ***3*** types of paths

Paths

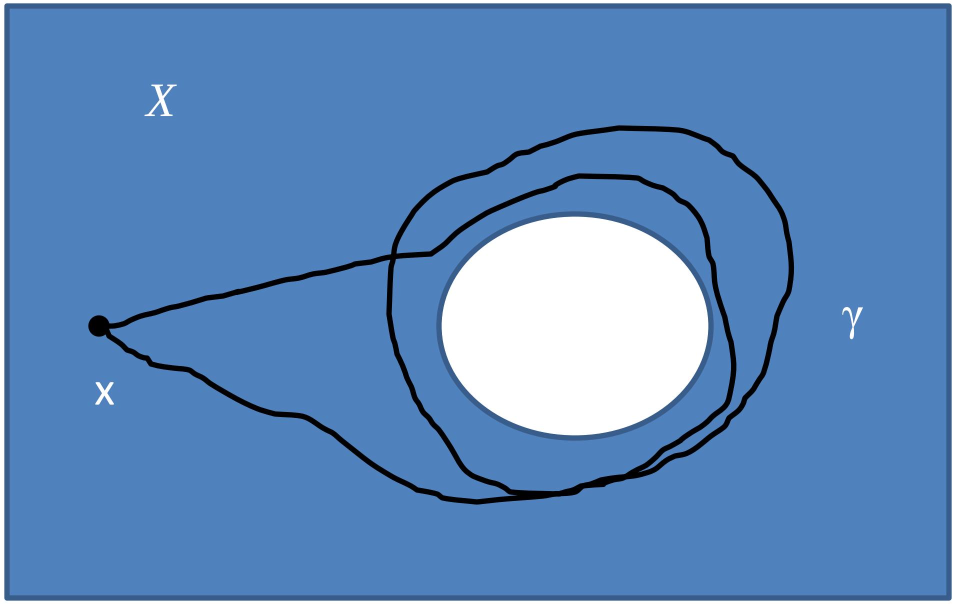


Path γ

$$I = [0,1] \rightarrow \gamma(t)$$

Are there "path types", *[γ]?*

Fundamental group



Path γ

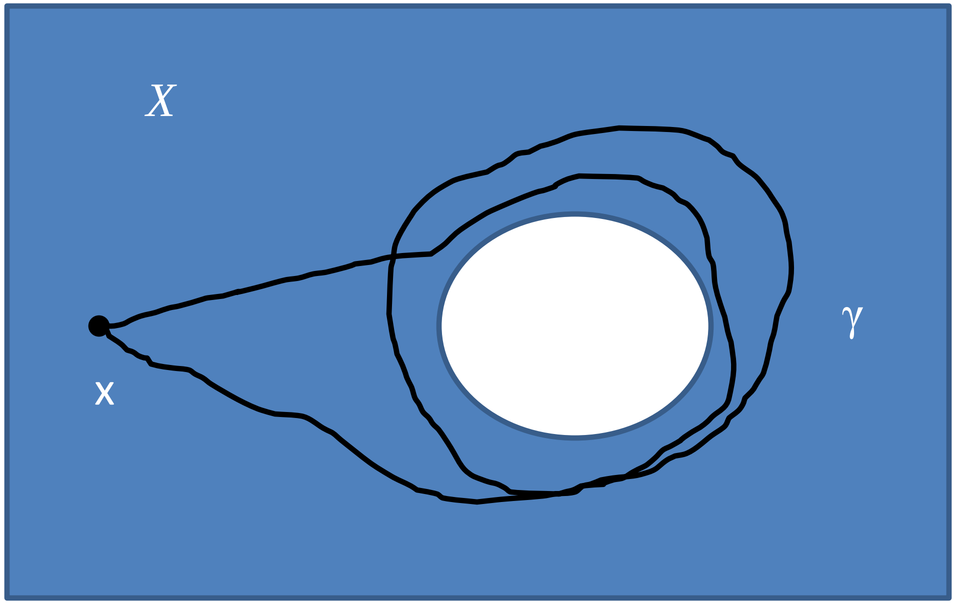
$$I = [0,1] \to \gamma(t)$$

1. Use closed paths $S^1 \to \gamma(t)$

Equivalence class,[γ], is defined, $\gamma \to [\gamma]_m$

Topological index m of the path $[\gamma]_m$

Fundamental group



1. *Use closed paths* $S^1 \to \gamma(t)$

2. *Paths combine:* $\gamma_1 \cdot \gamma_2 = \gamma_3$

3. *Equivalence classes combine:*

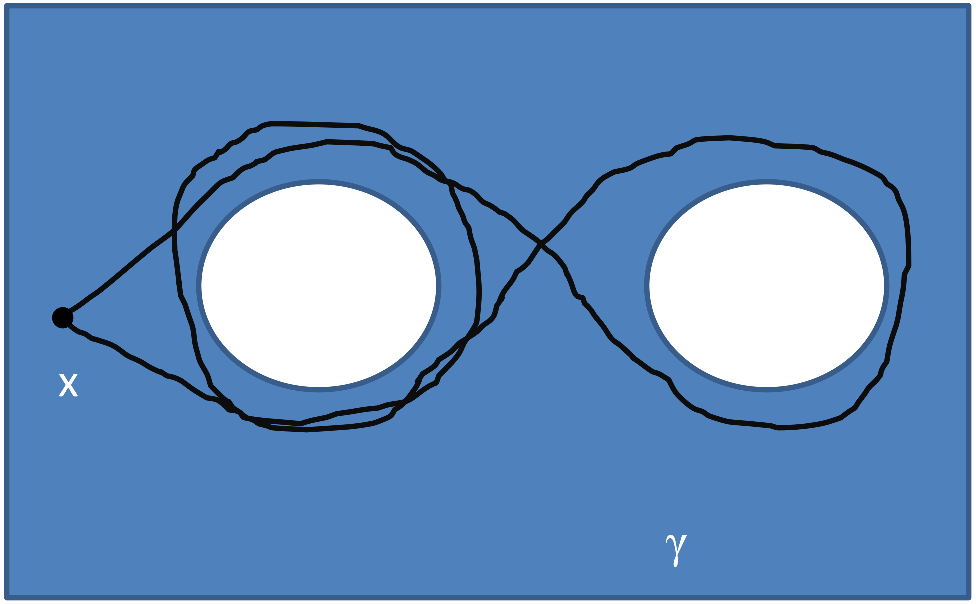
$$[\gamma_1]\cdot[\gamma_2]=[\gamma_3]$$

Topological index combines: $[\gamma_1]_m \cdot [\gamma_2]_n = [\gamma_3]_{n+m}$ *(n,m) →n+m*

$$(paths) \to (indexes)$$

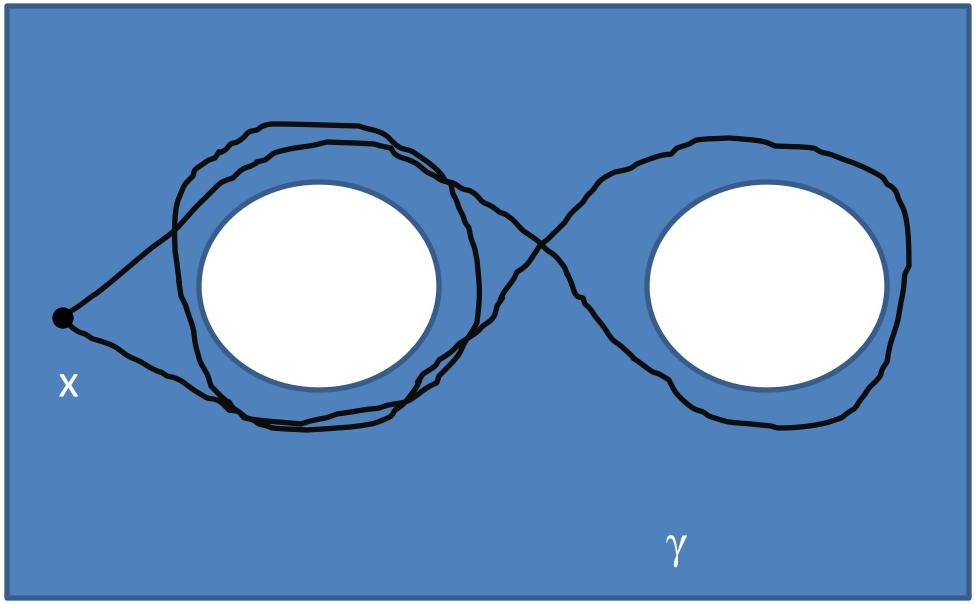
Fundamental group, $\pi_1(X) = \mathbb{Z}$

Two holes



$$\gamma \rightarrow [\gamma]_{(index?)}$$

Two holes

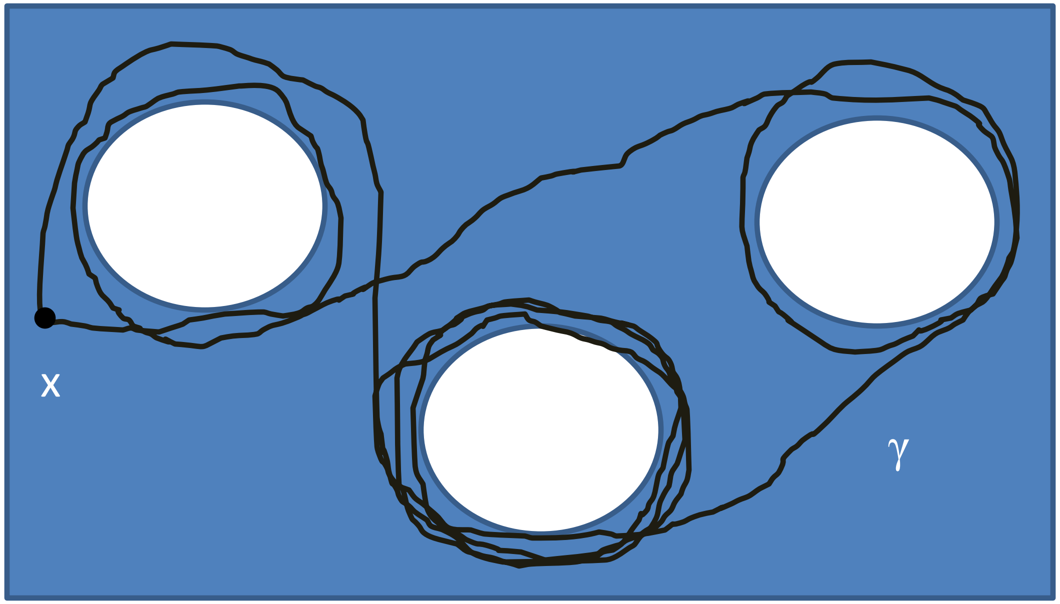


$$\gamma \to [\gamma]_{(m,n)}$$

$$[\gamma_1]_{(m_1,m_2)} \cdot [\gamma_2]_{(n_1,n_2)} = [\gamma_3]_{(m_1+n_1,m_2+n_2)}$$

Fundamental group, $\pi_1 = \mathbb{Z} \otimes \mathbb{Z}$

3 holes

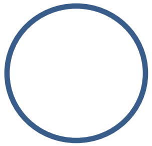


$$\gamma = [\gamma]_{(m,n,k)}$$

Fundamental group, $\pi_1 = \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z}$

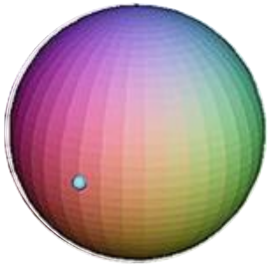
Fundamental group, examples

Circle



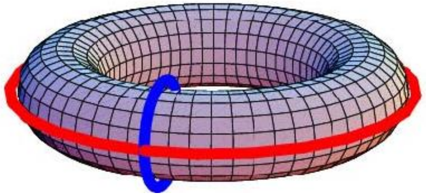
$$\pi_1 = \mathbb{Z}$$

Sphere



$$\pi_1 = 0$$

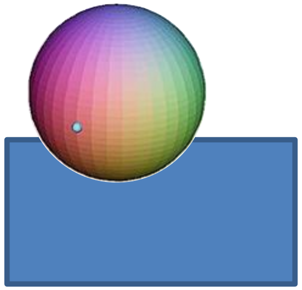
Torus



$$\pi_1 = \mathbb{Z} \otimes \mathbb{Z}$$

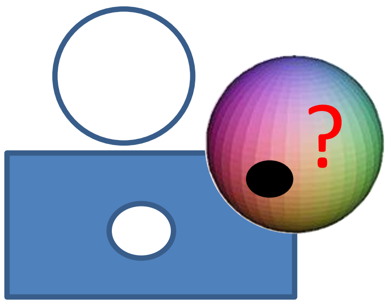
How useful are topological indexes?

$\pi_1 = 0$



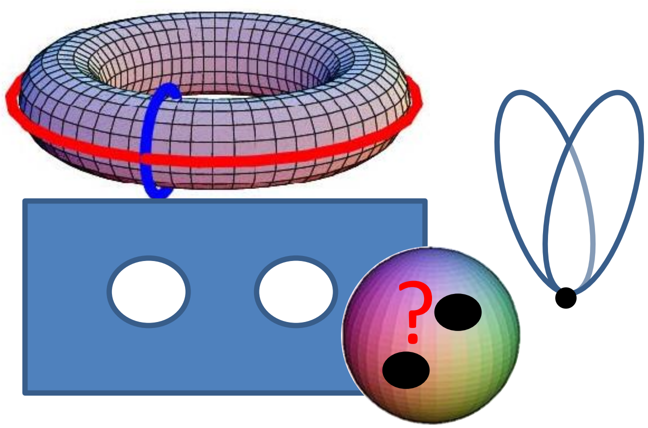
No loops

$\pi_1 = \mathbb{Z}$



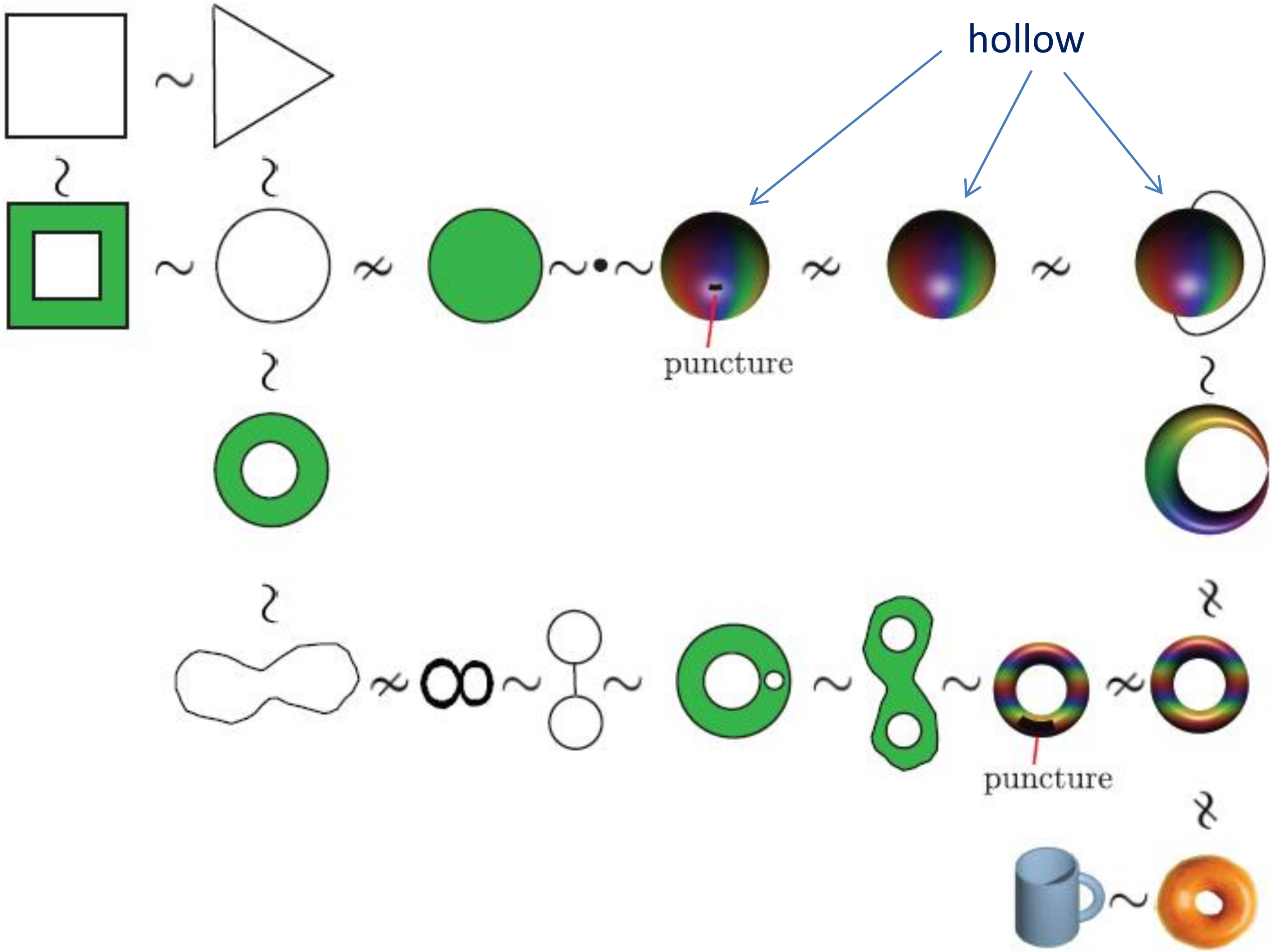
1 loop

$\pi_1 = \mathbb{Z} \otimes \mathbb{Z}$

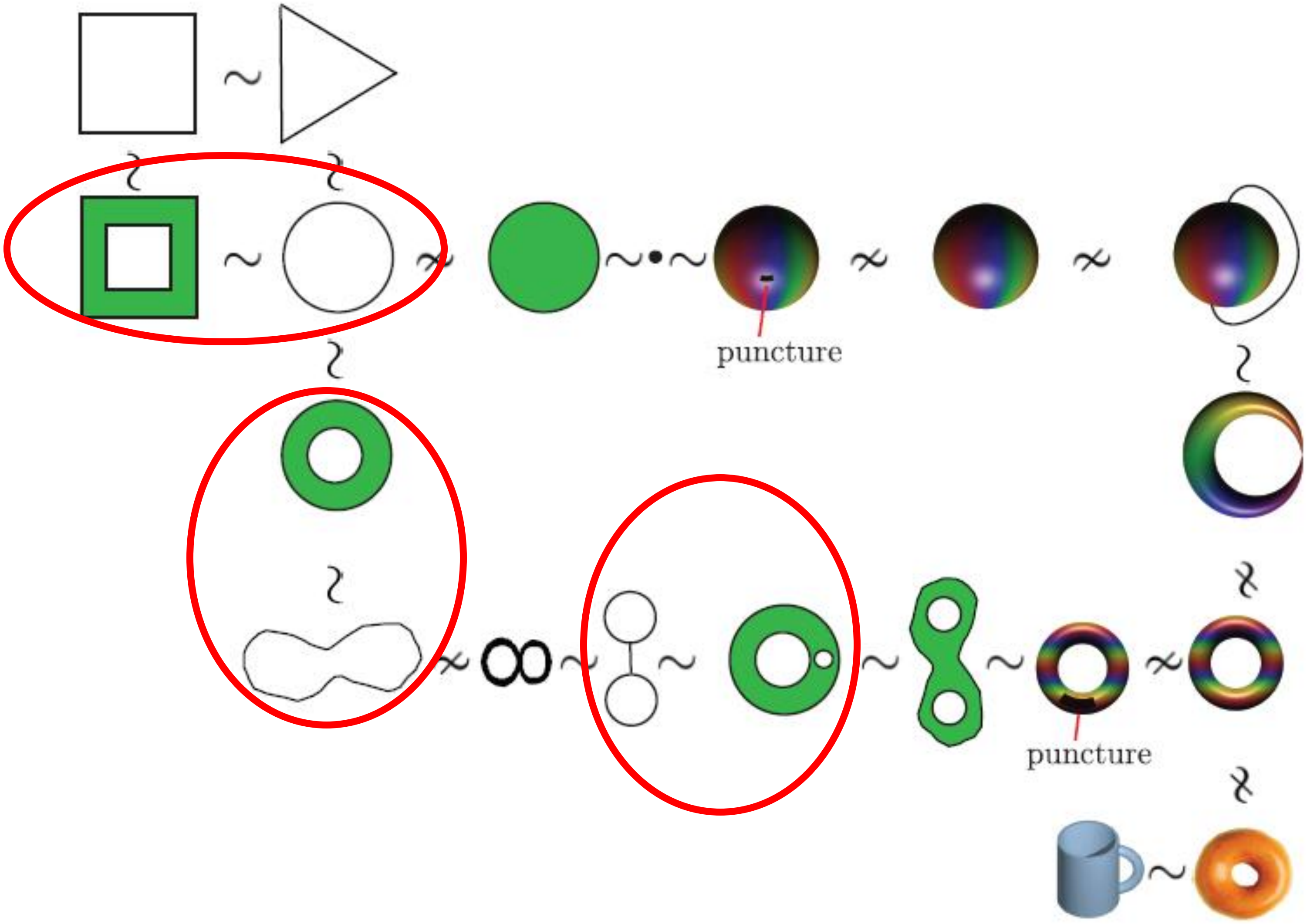


2 loops

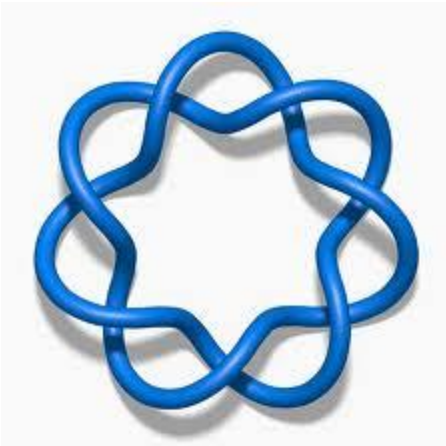
Topological equivalence and homotopies

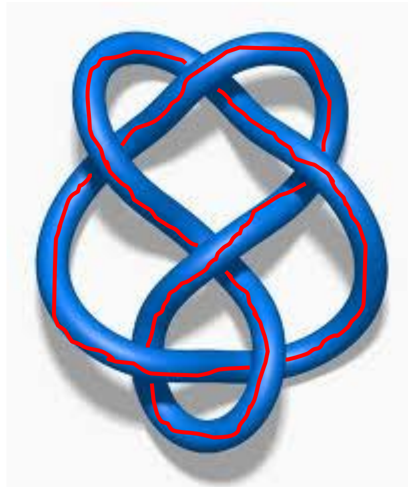


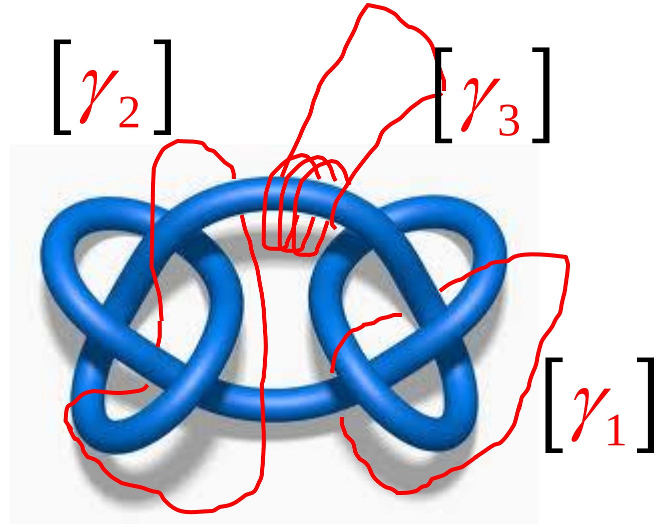
Topological equivalence and homotopies

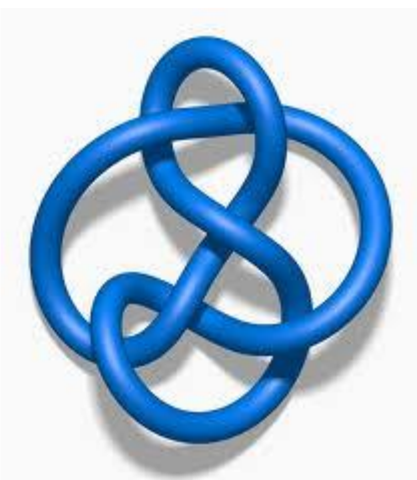


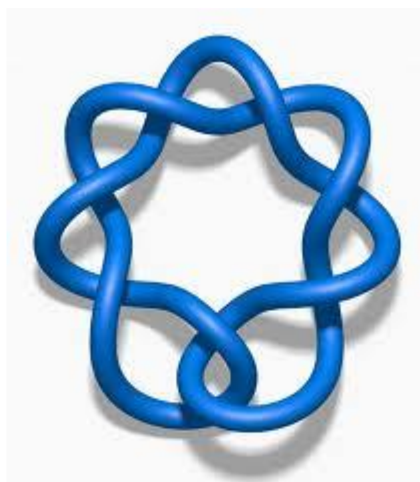
Applied Topology: knots



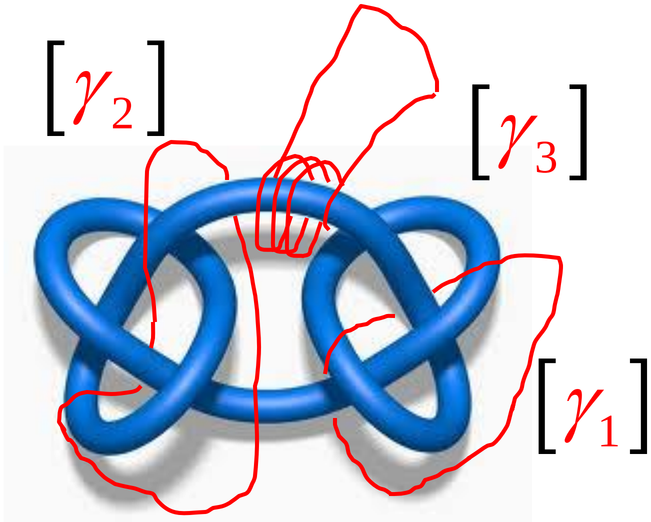


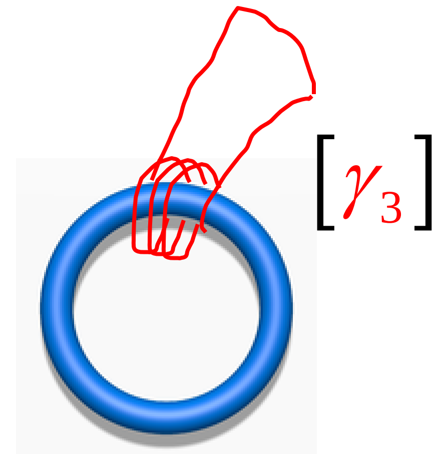




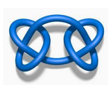


Applied Topology: embedding aspect



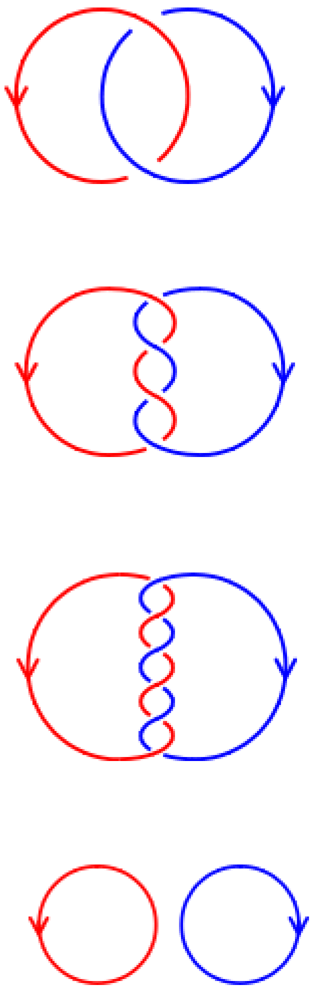


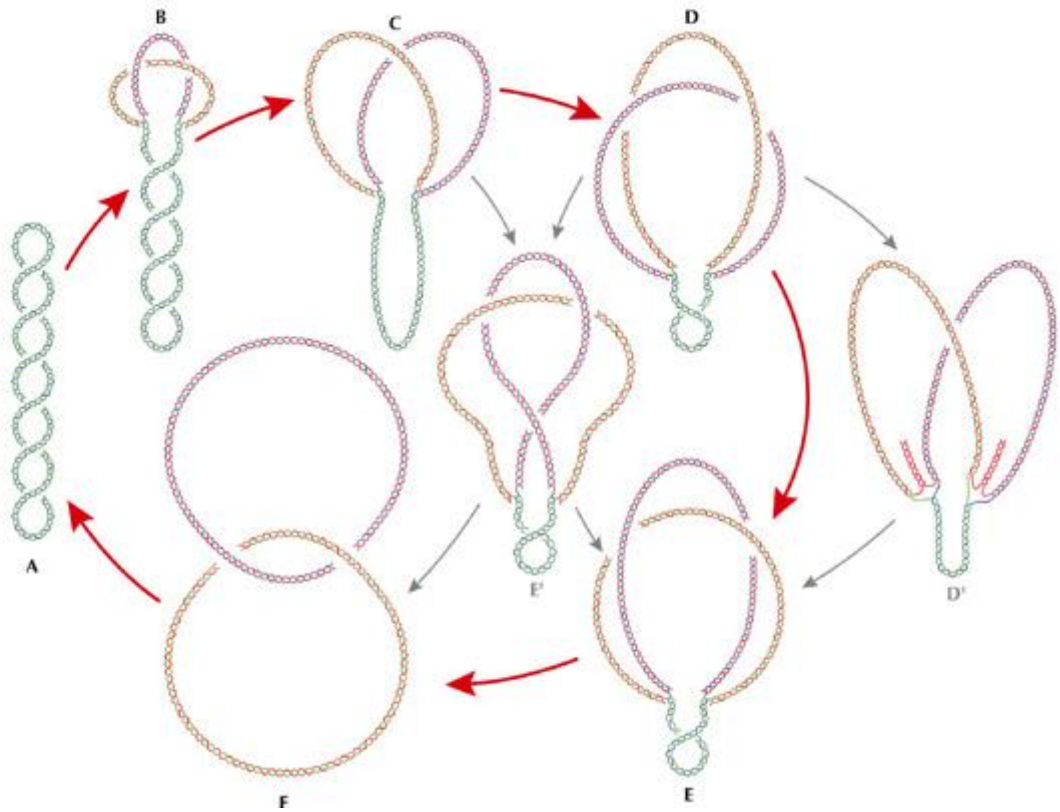
The fundamental group of the knot complements in $\boldsymbol{R}^3$ are different

$\pi_1(R^3\setminus$  $) \neq \pi_1(R^3\setminus$  $)$

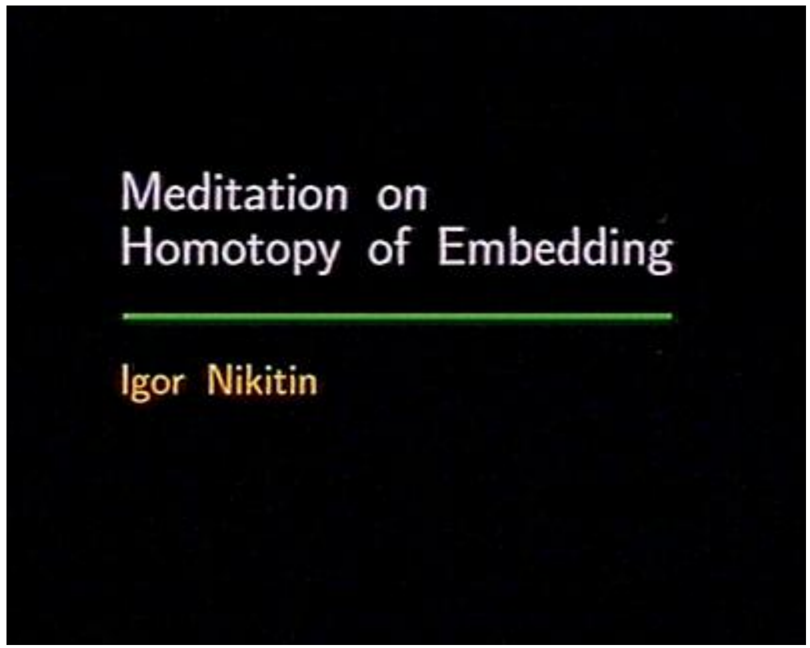
$\Longrightarrow$ knots are different

Applied Topology: links





The topological cycle of a replicon, EMBO reports (2004)


Meditation on
Homotopy of Embedding
Igor Nikitin

Summary

1. Topology, the idea
2. The notion of continuity
3. Topological indexes
4. Fundamental group

Next: Simplicial complexes and homologies

Literature:

1. J. Munkres, "*Topology*" (2000) and "*Elements Of Algebraic Topology*" (1996)
2. A. Zomorodian, "*Topology for computing*", (2009)
3. A. Hatcher, *"Algebraic Topology"*, (2002), http://www.math.cornell.edu/~hatcher/AT/ATpage.html